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Two-Dimensional Percolation and Classical String Theory

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Both two-dimensional percolation and classical string theory are shown to correspond to $c=0$ in the 2d conformal algebra. Therefore, despite being different theories, they have the same underlying conformal structure.

Two-dimensional conformal symmetry is an interesting approach to both field theories and statistical systems (Belavin *et al.*, 1984; Boyanovsky and Naon, 1987). It has been applied successfully to Ising models (Belavin *et al.*, 1984), to $1<q<4$ Potts models (Dotsenko and Fateev, 1984), to strings and superstrings (Friedan *et al.*, 1986) and to many other systems (Boyanovsky and Naon, 1987). Furthermore, it has related different models, e.g., the tricritical Ising model and supersymmetry (Qiu, 1986) and the superstring theory and the three-dimensional Ising model (Dotsenko, 1987; and Orlando, 1988). In this paper I will use conformal symmetry to relate 2d percolation theory to classical string theory.

First I summarize the results of this approach. The 2d conformal algebra is given by

$$[L_n, L_m]=(n-m)L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n+m_{10}} \tag{1}$$

where c is a c-number, called the parameter of the theory, which characterizes the theory being studied. There is a subset of the operators of this theory, called primary operators $\{\phi_n\}$, which under the conformal transformation

$$z\to\zeta(z), \qquad \bar{z}\to\bar{\zeta}(\bar{z})$$

transform according to the formula

$$\phi_n(z,\bar{z})\to\left(\frac{d\zeta}{dz}\right)^{\Delta_n}\left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\overline{\Delta_n}}\phi_n(\zeta,\bar{\zeta}) \tag{2}$$

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where Δ_n and $\overline{\Delta_n}$ are called the scale (conformal) dimensions of ϕ_n. They are related to its space-time dimension d_n and spin s_n by formulas

$$d_n = \Delta_n + \overline{\Delta_n}, \qquad s_n = \Delta_n - \overline{\Delta_n} \tag{3}$$

It has been shown (Belavin *et al.*, 1984) that all the correlation functions of the theory can be obtained from the primary field correlation functions.

Each primary field together with its secondary ones form a conformal family $[\phi_n]$ which forms a representation of the Virasoro algebra (1). If the dimensions Δ_n and $\overline{\Delta_n}$ are given by the Kac formulas

$$\begin{aligned} \Delta_{nm} &= \Delta_0 + \tfrac{1}{4}(n\alpha_+ + m\alpha_-)^2 \\ \Delta_0 &= \tfrac{1}{24}(c-1) \\ \alpha_\pm &= [(1-c)^{1/2} \pm (25-c)^{1/2}]/(24)^{1/2} \end{aligned} \tag{4}$$

where n and m are positive integers, then the representation $[\phi_n]$ is degenerate. Furthermore, if the parameter of the theory c satisfies the relation

$$\frac{(25-c)^{1/2}-(1-c)^{1/2}}{(25-c)^{1/2}+(1-c)^{1/2}} = \frac{P}{k} \tag{5}$$

where P and k are positive integers, then the theory is called minimal. In this case the theory has only a finite number of primary fields and all of them are degenerate. These are the interesting theories. It is conjectured (Boyanovsky and Naon, 1987) that these theories correspond to statistical systems which undergo second-order phase transitions. If one imposes unitarity, then two cases are possible (Friedan *et al.*, 1984). The first is $c>1$ and $\Delta_{nm} \geq 0$; in this case unitarity cannot restrict c or Δ_{nm} any further. The second is $c<1$, in which case c has to be in the form

$$c = 1 - 6/m(m+1), \qquad m = 2, 3, \ldots \tag{6}$$

and the corresponding conformal field dimensions are given by

$$\Delta_{pq} = \{[p(m+1) - q_m]^2 - 1\}/4m(m+1) \tag{7}$$

The value $c=\frac{1}{2}$ corresponds to the Ising model (Belavin *et al.*, 1984), $c=\frac{4}{5}$ corresponds to the three-state Potts model (Dotsenko and Fateev, 1984), and $c=26$ corresponds to the quantum theory of strings (Friedan *et al.*, 1986).

I will show that both 2d percolation theory and classical string theory correspond to $c=0$.

In string theory it is known that (Schwarz, 1982) the central charge c arises due to the normal ordering of the operators α_n^μ defined by

$$X^\mu(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu \, e^{in\tau} \cos n\sigma, \qquad \mu = 1, 2, \ldots, D \tag{8}$$

where $X^{\mu}(\sigma, \tau)$ are the space-time coordinates and σ and τ are world sheet coordinates parametrizing the string. For classical string theory, however, there is no normal ordering; hence, classical string theory corresponds to

$$c = 0 \tag{9}$$

This result makes classical string theory consistent in any space-time dimension, not necessarily $d = 26$.

To determine the parameter c for percolation theory, I use the result (Wu, 1982) that 2d bond percolation theory corresponds to $q = 1$ in the q-state Potts model. The conformal structure of this model has been studied (Dotsenko and Fateev, 1984) and the relation between q and P/k is

$$\begin{gathered} P/k \equiv (2N-1)/2N, \qquad y = 1/N \\ \sqrt{q} = 2\cos(\pi y/2) \end{gathered} \tag{10}$$

Therefore I conclude that 2d percolation theory corresponds to $c = 0$. This result has been obtained (Saleur, 1987) by studying the finite-size corrections to the free energy.

Therefore I conclude that both 2d percolation and classical string theory correspond to $c = 0$, i.e., they have identical conformal structure. This confirms the importance of the conformal symmetry approach to both statistical systems and field theories.

Unfortunately, the formulas (4) and (7) which eventually give the critical exponents of the system do not work for $c = 0$. However, the method of the transfer matrix has been used (Saleur, 1987) to obtain the critical exponents for the 2d bond percolation problem. The invariance of the critical exponents in some percolation problems under conformal transformations has been shown (Ahmed and Tawansi, 1988).

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